Let $x := \lim x_n$, $y := \lim y_n$ s.t. $y, y_n \neq 0 \ \forall n$.

Let $\varepsilon > 0$ be given. Take $\varepsilon_1, \varepsilon_2 > 0$ accordingly:

$$\varepsilon_1 := \frac{\varepsilon}{2} \cdot \frac{|y|}{2} \quad \text{(or smaller positive)}$$

$$\varepsilon_2 := \min\left\{ \frac{|y|}{2}, \ \frac{\varepsilon}{2} \cdot \frac{|y|^2}{2} \cdot \frac{1}{|x|+1} \right\}$$

Let $N_1, N_2 \in \mathbb{N}$ be s.t.

$$|x_n - x| < \varepsilon_1 \quad \forall n \geq N_1$$

$$|y_n - y| < \varepsilon_2 \quad \forall n \geq N_2$$

Let $n \geq N_1 \vee N_2$. To show

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < \varepsilon. \qquad (*)$$

Note that $|y| - |y_n| \leq |y - y_n| < \varepsilon_2 \leq \frac{|y|}{2}$ so $\frac{|y|}{2} < |y_n|$ & $\frac{1}{|y_n|} < \frac{2}{|y|}$

and

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{x_n y - y_n x - xy + xy}{|y_n| \cdot |y|} \right| \leq \frac{2}{|y| \cdot |y|} \left(|y| \cdot |x_n - x| + |x|\,|y_n - y| \right)$$

$$< \frac{2}{|y|^2} \left(|y| \cdot \varepsilon_1 + |x|\, \varepsilon_2 \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because $\varepsilon_1 \leq \frac{\varepsilon}{2} \cdot \frac{|y|}{2}$ and $\varepsilon_2 \leq \frac{\varepsilon}{2} \cdot \frac{|y|^2}{2} \cdot \frac{1}{|x|+1}$

(used $|x|+1$ because x may be zero).

Question. If I take

$$\varepsilon_2 := \min\left\{ \frac{2|y|}{3}, \ \ldots \right\}$$

$$\varepsilon_1 := \ldots\ldots$$

当然 s. ε 有关

can you fill the blanks and check?

Similarly can you do product rule along the same line: Let $x := \lim x_n$, $y = \lim y_n$.
Let $\varepsilon > 0$. Take $\varepsilon_1, \varepsilon_2 > 0$ s.t.

$$\varepsilon_1 := \min\left\{1, \frac{\varepsilon/2}{|y|+1}\right\},$$

$$\varepsilon_2 := \frac{\varepsilon/2}{|x|+1} \quad \text{(or smaller \& positive)}$$

and take $N_1, N_2 \in \mathbb{N}$ as before corresponding to $\varepsilon_1, \varepsilon_2$. Let $n \geq N_1, N_2$. To show

$$|x_n y_n - xy| < \varepsilon \qquad (**)$$

Note that $|y_n| - |y| \leq |y_n - y| < \varepsilon_1 \leq 1$ and

$$|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| \leq |y_n| |x_n - x| + |x| \cdot |y_n - y|$$

$$< (|y|+1)\varepsilon_1 + (|x|+1)\varepsilon_2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because $\varepsilon_1 \leq \dfrac{\varepsilon/2}{|y|+1}$ and $\varepsilon_2 = \dfrac{\varepsilon/2}{|x|+1}$.

$\therefore$ $(**)$ holds.